SNR Calculation and Spectral Estimation [S&T Appendix A]

or, How *not* to make a mess of an FFT

0 Make sure the input is located in an FFT bin

1 Window the data!

A Hann window works well.

2 Compute the FFT

3 SNR = power in signal bins / power in noise bins

4 If you want to make a spectral plot

- **i. Apply sine-wave scaling**
- **ii. State the noise bandwidth (NBW)**
- **iii. Smooth the FFT**

FT and DFT (1)

- **Fourier Transform:**

$$x(t) \leftrightarrow X(\omega)$$

- **If** $x(t)$ **is sampled**

$$x(nT) \leftrightarrow \sum_{n=-\infty}^{\infty} x(nT) \cdot e^{-j\omega nT}$$

- **Estimation of spectrum: DFT / FFT**

$$X(f_k) = \sum_{n=0}^{N-1} x(nT) \cdot e^{-j2\pi nTf_k} = x(nT) * h_k(n)$$

where $f_k = k/(NT), \quad k = 0,\ 1,\ 2,\ \ldots,\ N\text{-}1$

$$h_k(n) = \begin{cases} e^{j2\pi kn/N} & ,0 \le n \le N \\ 0 & ,\text{otherwise} \end{cases}$$

FT and DFT (2)

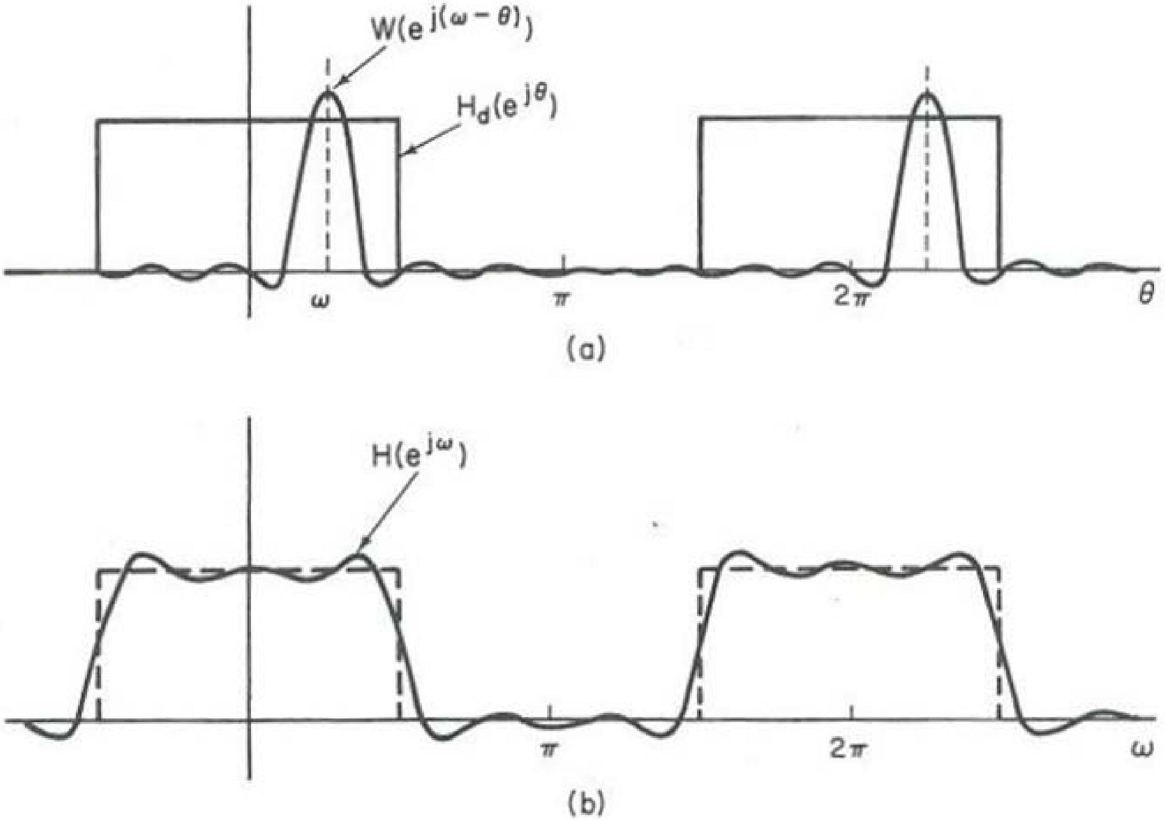
- **Generally, in Fourier Transformation, the rule is**

$$\boxed{Sampled \ \leftrightarrow \ Periodic}$$

- **If $x(nT)$ is <u>not</u> periodic with period *NT*, the DFT calculates the spectrum of a discontinuous signal -- bad estimate!**

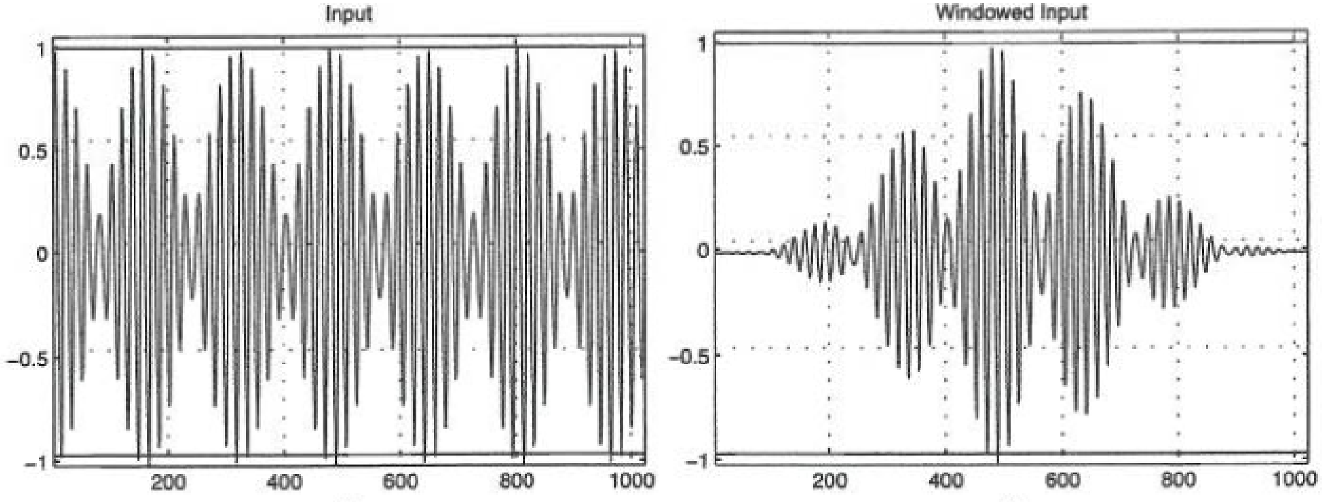
FT and DFT (3)

- **Another problem: convolution introduces noise folding in windowed spectrum:**



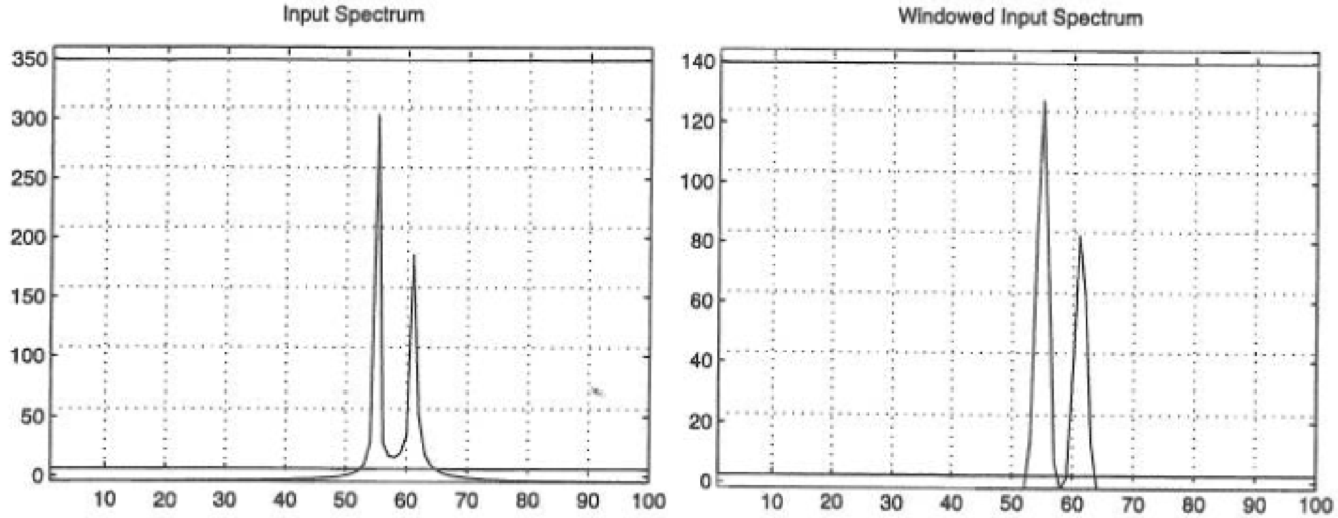
(a) Convolution process implied by truncation of the ideal impulse response. (b) Typical approximation resulting from windowing the ideal impulse response.

Example: two-tone signal



(a) (b)

Time domain input of the example before (a) and after windowing (b).



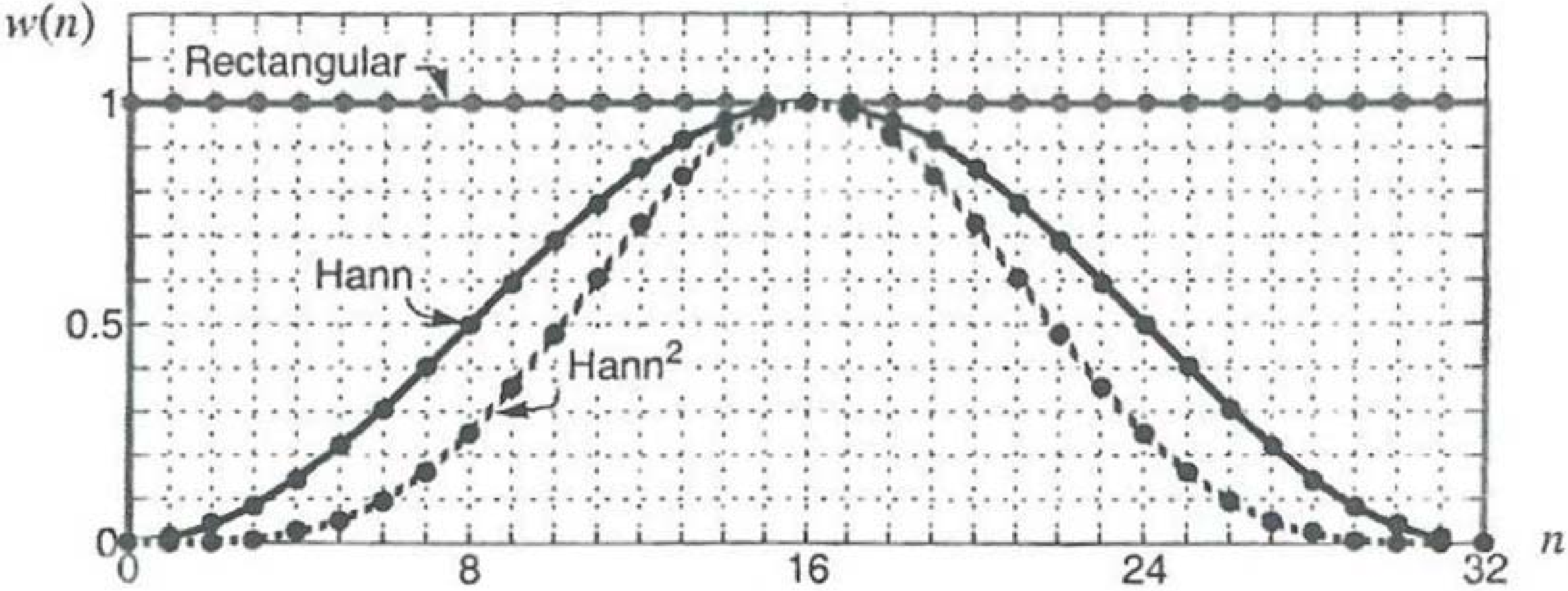
Spectra with and without windowing.

Windowing

- **General window (T=1):**

$$W(f) = \sum_{n=0}^{N-1} w(n) \cdot e^{-j2\pi fn}$$

- **Common windows:**



Windowing

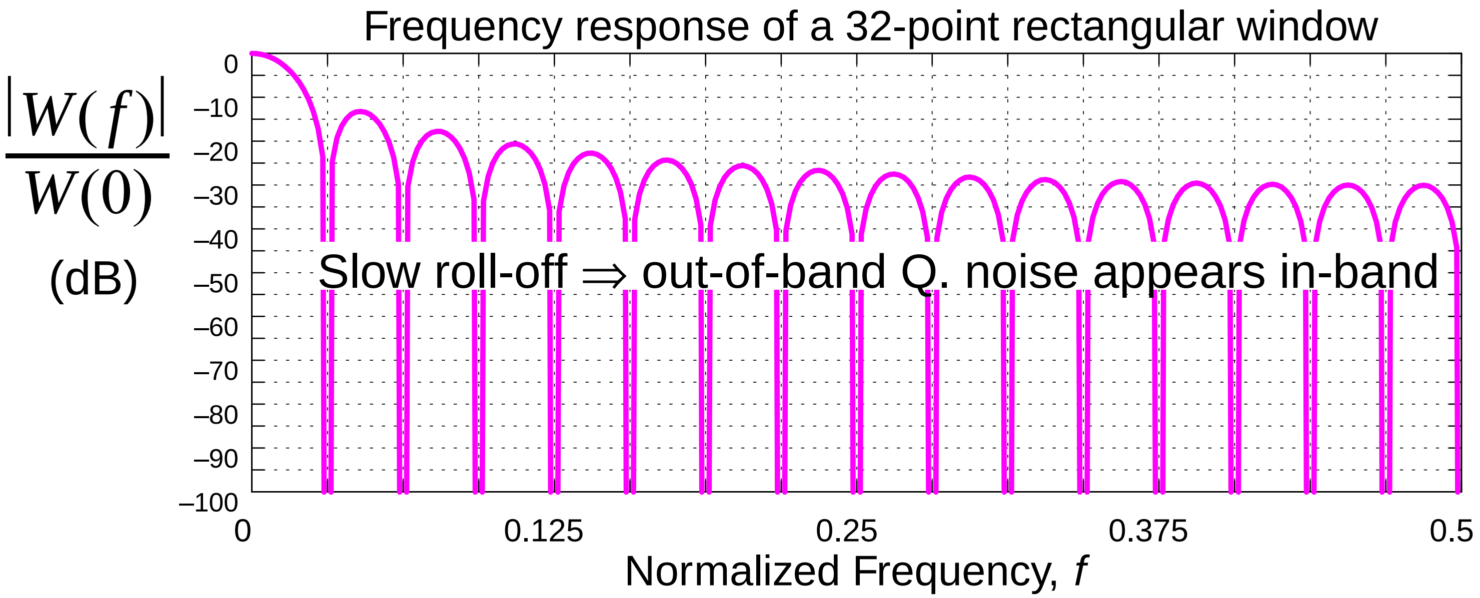
- **ΔΣ data is (usually) not periodic**

  Just because the input repeats does not mean that the output does too!

- **A finite-length data record = an infinite record multiplied by a *rectangular window*: $w(n) = 1, 0 \le n < N$**

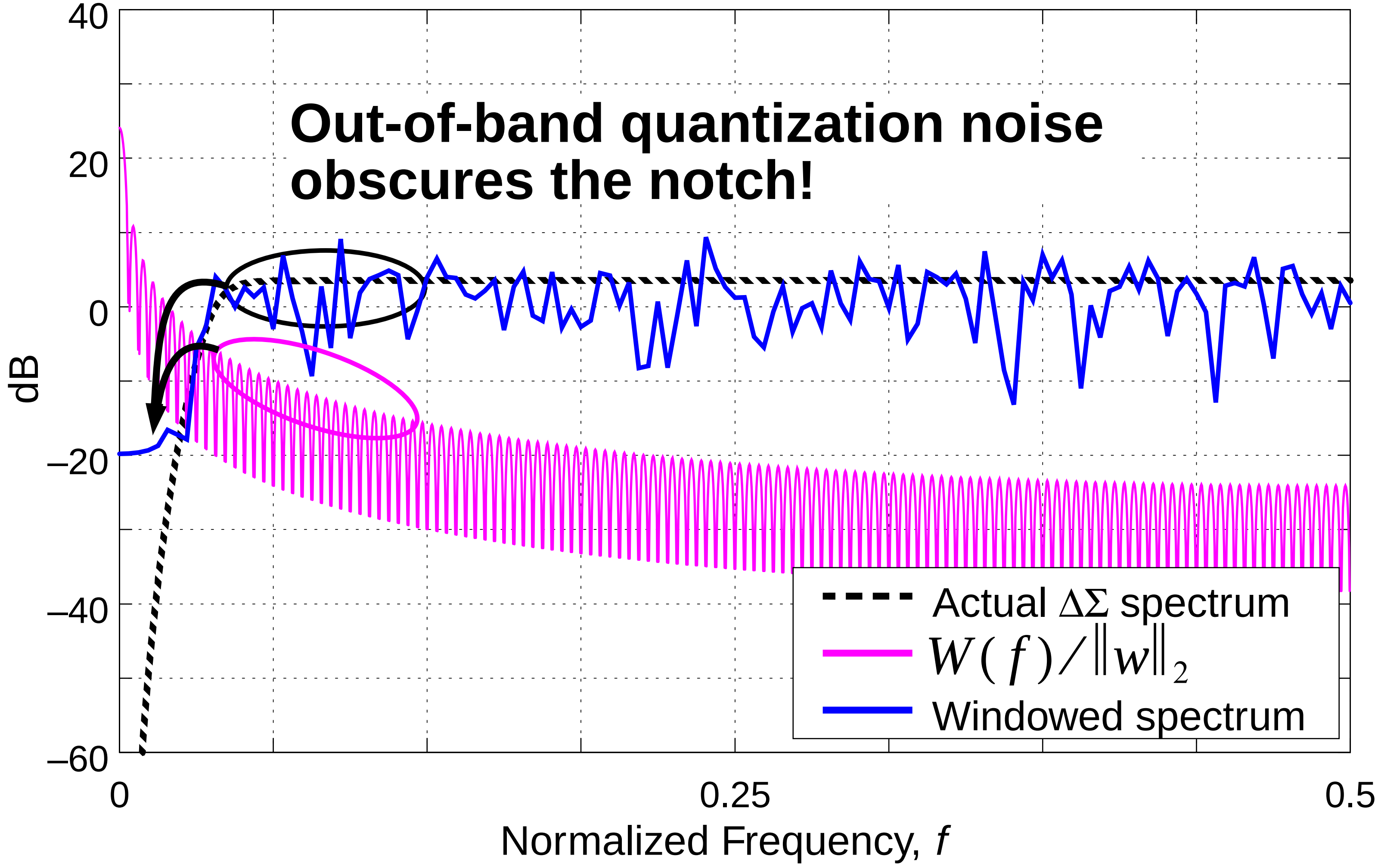
  Windowing is unavoidable.

- **"Multiplication in time is convolution in frequency"**



Example Spectral Disaster

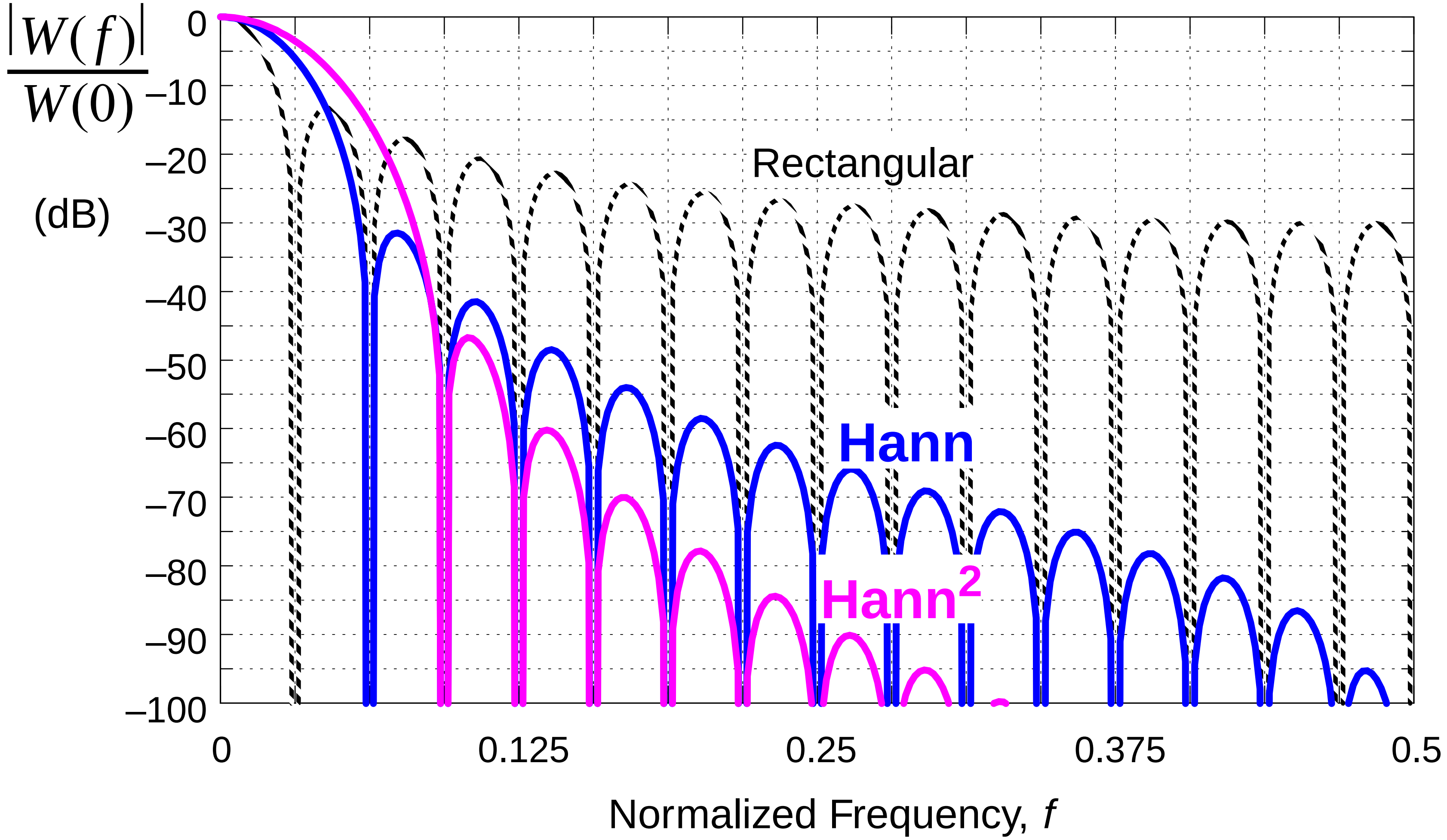
Rectangular window, *N* = 256



Out-of-band quantization noise obscures the notch!

Window Comparison

$N = 16$



Window Properties

Window	Rectangular	Hann‡	Hann^2
$w(n)$, $n = 0, 1, \ldots, N-1$ ($w(n) = 0$ otherwise)	**1**	$\dfrac{1-\cos\frac{2\pi n}{N}}{2}$	$\left(\dfrac{1-\cos\frac{2\pi n}{N}}{2}\right)^2$
Number of non-zero FFT bins	**1**	**3**	**5**
$\Vert w\Vert_2^2 = \sum w(n)^2$	N	$3N/8$	$35N/128$
$W(0) = \sum w(n)$	N	$N/2$	$3N/8$
$NBW = \dfrac{\Vert w\Vert_2^2}{W(0)^2}$	$1/N$	$1.5/N$	$35/18N$

‡. MATLAB's `hanning` function causes spectral leakage of tones located in FFT bins unless you add the optional argument "`periodic`". Use ΔΣ Toolbox function ds_hann.

Window Length, *N*

- **Need to have enough in-band noise bins to**
  1. **Make the number of signal bins a small fraction of the total number of in-band bins**

     <20% signal bins ⇒ >15 in-band bins ⇒ $N > 30 \cdot OSR$
  2. **Make the SNR repeatable**

     $N = 30 \cdot OSR$ yields std. dev. ~1.4 dB.

     $N = 64 \cdot OSR$ yields std. dev. ~1.0 dB.

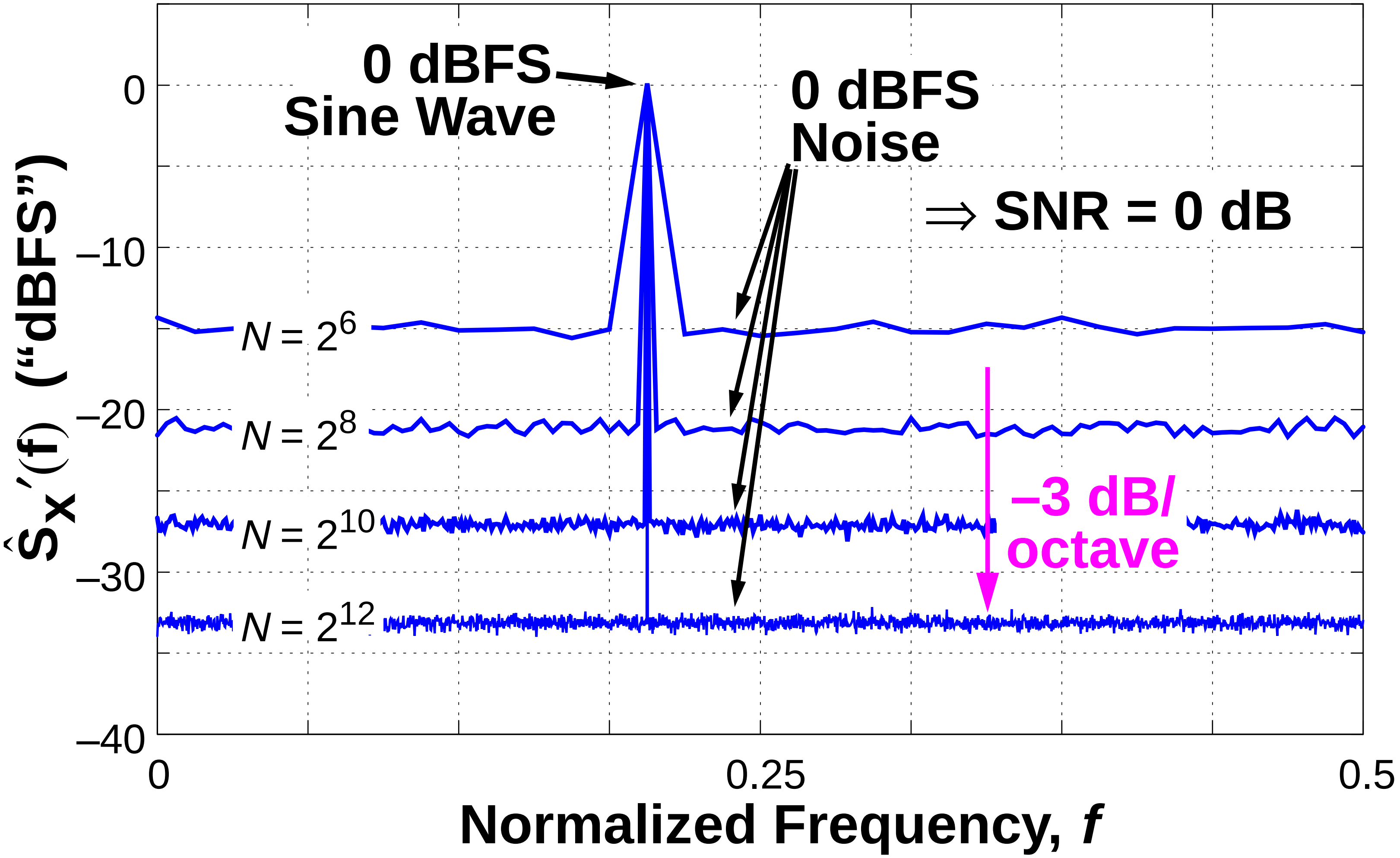
     $N = 256 \cdot OSR$ yields std. dev. ~0.5 dB.

- **$N = 64 \cdot OSR$ is recommended**

This is all you need to know to do SNR calculations.
If you want to make spectral plots, you need to know more…

Spectrum of a Sine Wave plus Noise

Various *N*



Scaling and Noise Bandwidth

- **FFT scaled such that a full-scale (FS) sine wave ($A = FS/2$) yields a 0-dB spectral peak:**

$$\hat{S}_x'(f) = \left| \frac{\sum_{n=0}^{N-1} w(n) \cdot x(n) \cdot e^{-j2\pi fn}}{(FS/4)W(0)} \right|^2 \begin{array}{l} \longleftarrow |\mathrm{FFT}|^2 \\ \\ \longleftarrow \text{sine-wave scale factor} \end{array}$$

- **"Noise Floor" depends on N (!)**

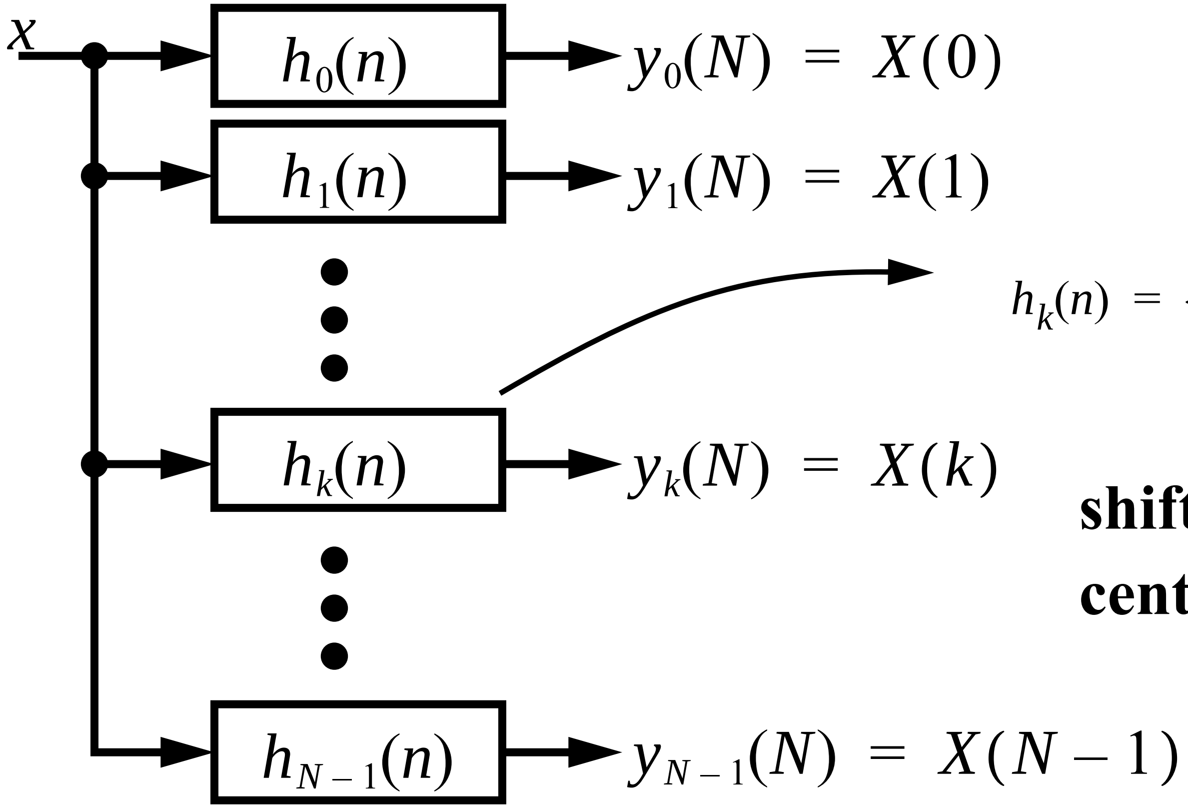
  A sine-wave-scaled FFT is fine for showing *spectra*, but is ill-suited for displaying *spectral densities*.

- **Vertical axis is really "dBFS/NBW," where NBW is the bandwidth over which the noise power has been integrated**

  Think of the spectrum as representing the amount of power in a frequency band whose width is NBW.
  $NBW = k/N$, where k depends on the window type.

An FFT is like a Filter Bank

- **The FFT can be interpreted as taking 1 sample from the outputs of *N* complex FIR filters:**



x

$h_0(n)$ → $y_0(N) = X(0)$

$h_1(n)$ → $y_1(N) = X(1)$

$h_k(n)$ → $y_k(N) = X(k)$

$h_{N-1}(n)$ → $y_{N-1}(N) = X(N-1)$

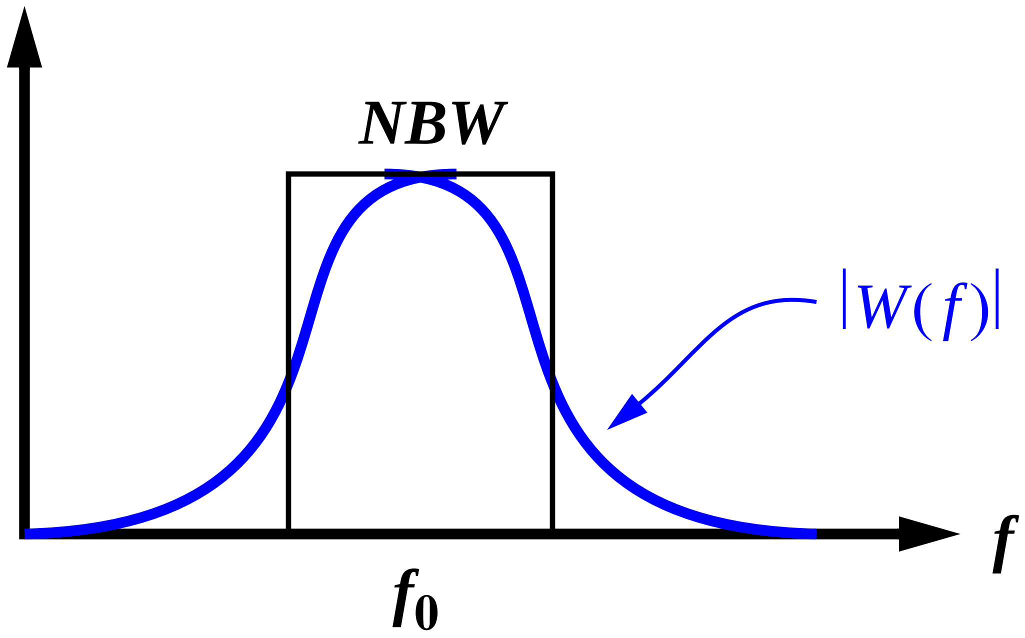
$$h_k(n) = \begin{cases} e^{j\frac{2\pi k}{N}n}, & 0 \le n < N \\ 0, & \text{otherwise} \end{cases}$$

shifted sinc functions, centered at $f_k = k/NT$

- **NBW is the effective bandwidth of these filters**

Noise Bandwidth

- **For a filter with frequency response $W(f)$**



$$NBW = \frac{\int |W(f)|^2 df}{W(f_0)^2}$$

Noise Bandwidth of a Rectangular FFT

$$h_k(n) = \exp\left(j\frac{2\pi k}{N}n\right)$$

$$W_k(f) = \sum_{n=0}^{N-1} h_k(n)\exp(-j2\pi fn)$$

$$f_0 = \frac{k}{N},\ W_k(f_0) = \sum_{n=0}^{N-1} 1 = N$$

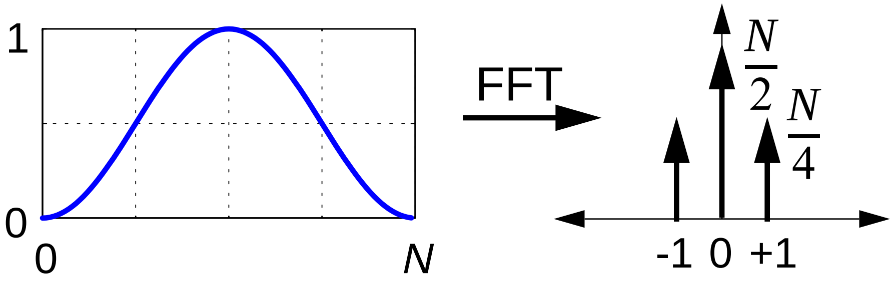
$$\int \left|W_k(f)\right|^2 = \sum \left|w_k(n)\right|^2 = N \qquad \textbf{[Parseval]}$$

$$\therefore NBW = \frac{\int \left|W_k(f)\right|^2 df}{W_k(f_0)^2} = \frac{N}{N^2} = \frac{1}{N}$$

- **NBW is the same for each FFT bin “filter”**

Noise Bandwidth of a Hann-Windowed FFT

- **Use the filter associated with FFT bin 0**
- $w(n) = \dfrac{1 - \cos\left(\dfrac{2\pi}{N} n\right)}{2}$



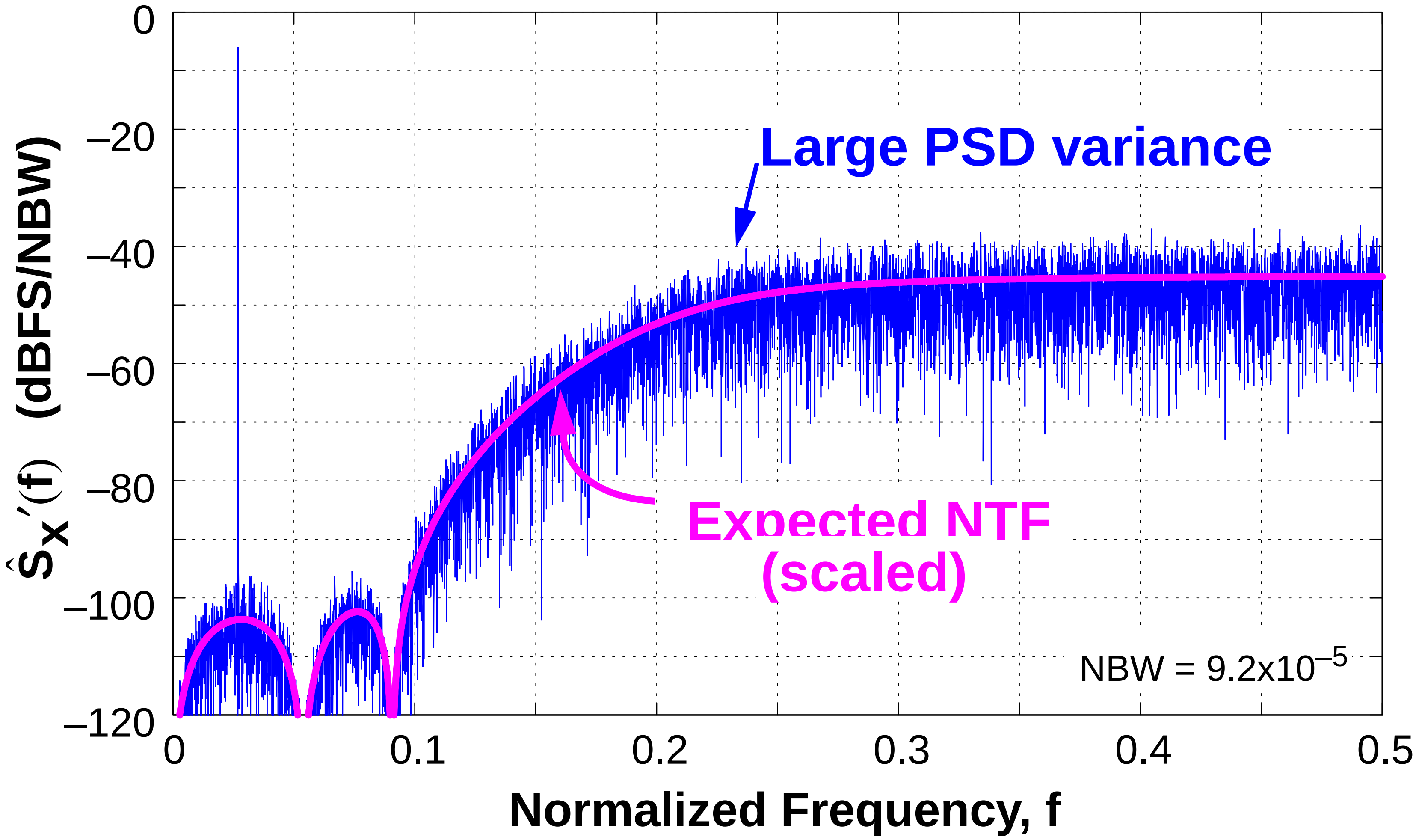
$$\Rightarrow \|w\|_2^2 = \frac{3N}{8},\ W(0) = \frac{N}{2}$$

$$\Rightarrow NBW = \frac{\|w\|_2^2}{W(0)^2} = \frac{1.5}{N}$$

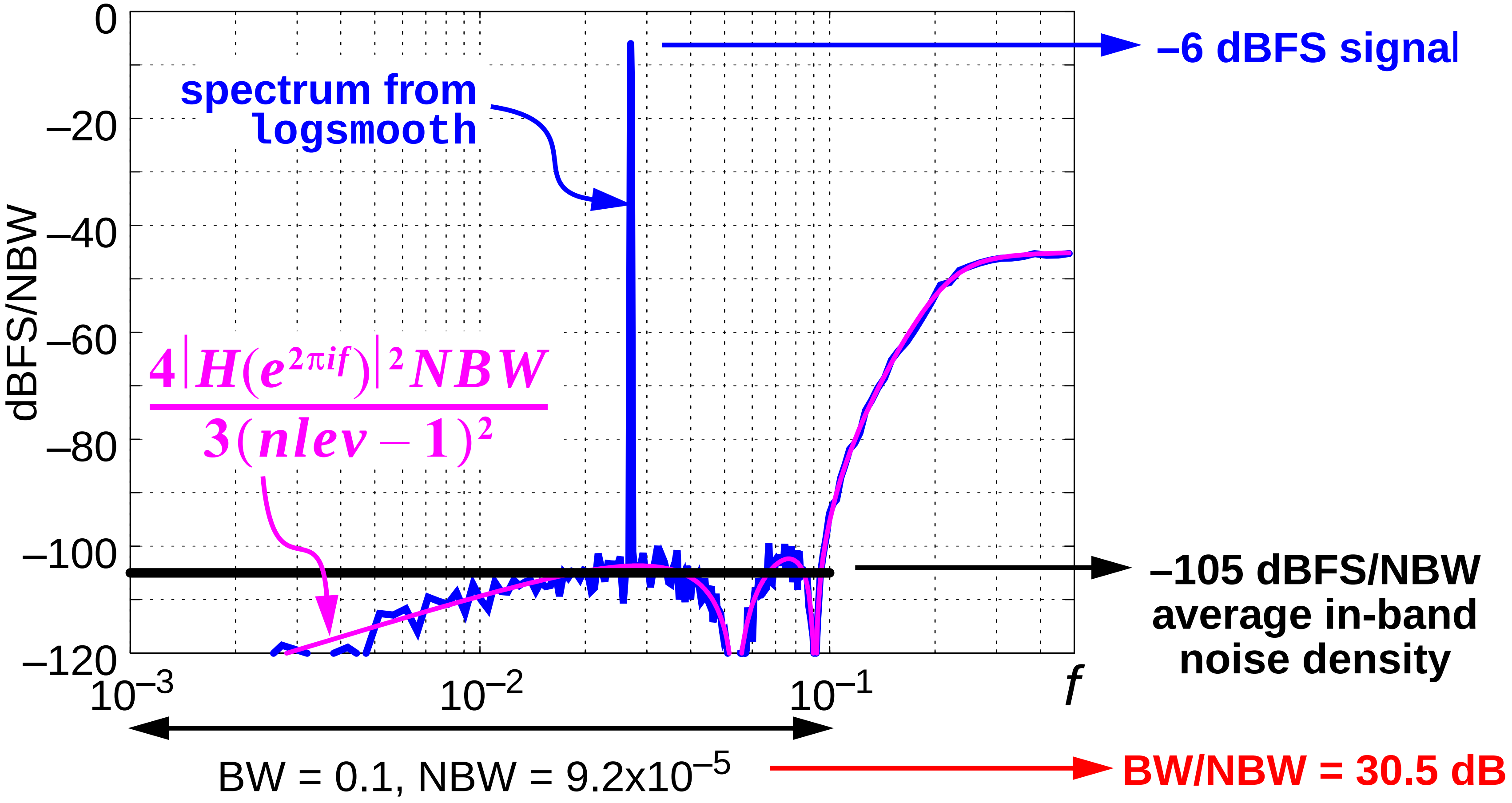
Three shifted sinc functions added.

Example ΔΣ Spectrum

5th-order NTF, OSR=5, Hann window, no averaging



Smoothed Spectrum and SNR Calculation



spectrum from `logsmooth`

$$\frac{4\left|H(e^{2\pi i f})\right|^2 NBW}{3(nlev-1)^2}$$

–6 dBFS signal

–105 dBFS/NBW
average in-band
noise density

BW = 0.1, NBW = 9.2×10^{-5}

BW/NBW = 30.5 dB

- **Quantization Noise Power = –105 + 30.5 = –74.5 dBFS**
  ⇒ SQNR = –6 – (–74.5) = 68.5 dB

Manual SQNR Prediction

- **The noise term is HE.**
- **The rms value of H in the band of interest, σ_H, can be evaluated using `rmsGain`.**
- **Since $\Delta = 2$ for all quantizers, $\sigma_e^2 = \frac{\Delta^2}{12} = \frac{1}{3}$.**
- **The in-band noise power is therefore $\frac{\sigma_H^2 \sigma_e^2}{\text{OSR}} = \frac{\sigma_H^2}{3\text{OSR}}$.**
- **The signal power is impossible to predict using the linear model, but is usually around –3 dBFS.**

  This corresponds to a power of $(nlev - 1)^2 / 4$.

- $\therefore$ $\text{SQNR}_{\text{peak}} \approx 10\log_{10}\left(\frac{3(\text{OSR})(nlev-1)^2}{4\sigma_H^2}\right)$ **dB.**